PROBLEM ON 2009 OCTOBER 26

MVHS NUMBER THEORY GROUP

A positive integer is called *perfect* if it is equal to the sum of all its proper divisors including 1. For example, 6 is a perfect number because the proper divisors of 6 are $1, 2, 3$ and we have that

$$6 = 1 + 2 + 3$$

The next perfect number is 28. Since $28 = 2^2 \cdot 7$, the divisors of 28 are $1, 2, 4, 7, 14$ and we have that

$$28 = 1 + 2 + 4 + 7 + 14$$

The next perfect number is 496. Since $496 = 2^4 \cdot 31$, the divisors of 496 are $1, 2, 4, 8, 16, 31, 62, 124, 248$ and we have that

$$496 = 1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248$$

Can you think of any other perfect numbers? By writing the first three perfect numbers slightly differently we can discern a certain pattern.

$$\begin{aligned} 6 &= 2^1 \cdot 3 && = 2^1 \cdot \left(2^2 - 1\right) = 1 + 2 + 3 \\ 28 &= 2^2 \cdot 7 && = 2^2 \cdot \left(2^3 - 1\right) = 1 + 2 + 4 + 7 + 14 \\ 496 &= 2^4 \cdot 31 && = 2^4 \cdot \left(2^5 - 1\right) = 1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248 \end{aligned}$$

Can you guess at what the pattern might be from these three specific examples? In all three cases the perfect number (call it P) is written in the form

$$P = 2^{p-1} \cdot (2^p - 1)$$

where p is a prime number. In our first three cases the values for p are $2, 3, 5$. What happens when $p = 7$? Is the number

$$P = 2^6 \cdot \left(2^7 - 1\right) = 8128$$

a perfect number? It turns out that it is. Verify that 8128 is indeed perfect. It is NOT true that for every prime number p the number P is perfect. For instance for $p = 11$ the number

$$P = 2^{10} \cdot \left(2^{11} - 1\right)$$

is not a perfect number. So what is the deal? for which primes p is $2^{p-1} \cdot (2^p - 1)$ a perfect number? To answer this examine the numbers $2^p - 1$ for $p = 2, 3, 5, 7, 11$. What is different about the number $2^{11} - 1$? Figuring out this part is worth **1 Point**. After you have figured this out conjecture about how the number $2^p - 1$ relates to the number P being perfect. Can you explain why this certain property about $2^p - 1$ always makes P a perfect number? This explanation is worth **2 Points**.

As some added commentary, the above problem gives a way to find numbers which we know are perfect. The converse statement of finding *all* perfect numbers is a problem that is still quite open to the mathematical community. All perfect numbers of the form

$$2^{p-1} \cdot (2^p - 1)$$

are obviously divisible by 2 and are thus even. No one knows if an odd perfect number exists.